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The Balanced-Projective Dimension of p-Local Abelian Groups

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By generalizing Hill's theorem giving a necessary and sufficient condition for an isotype subgroup of a totally projective p-group to be itself totally projective, we are able to give a condition which will determine the balanced-projective dimension of an arbitrary p-local abelian group when that dimension is finite. We rely on our recent third axiom of countability characterization for p-local balanced projectives and are able to carry over almost routinely the arguments given recently by Fuchs and Hill which were used to determine the balanced-projective dimension of an arbitrary abelian p-group. We are then able to prove that the balanced-projective dimension of any countable p-local abelian group and of any p-local Warfield group is less than or equal to 1. © 1987 Academic Press, Inc.

In a recent paper [1], Fuchs and Hill have obtained definitive results on the balanced-projective dimension of abelian p-groups. Their work depends heavily on techniques developed earlier by Hill in [3] and the main result of this latter paper also plays a crucial role in [1]. Although our original intention had been to apply our recent third axiom of countability characterization of p-local balanced projectives [6] merely to generalize Hill's theorem [3] on when an isotype subgroup of a totally projective is itself totally projective, the appearance of [1] lends added significance to this generalization and we are now able to extend to arbitrary p-local groups the characterization given in that paper of groups of balanced-projective dimension n for each $n < \omega$. The ease with which we are able to carry out this extension is, in large measure, attributable to the fact that [1] was obviously written in a fashion to accommodate such generalization. Another recent paper by Hill and Megibben [4] suggests that our generalization of the main result of [3] will also play a future role in the study of isotype subgroups of p-local balanced projectives.

Throughout this paper, we will be dealing with p-local groups; that is, with modules over the ring Z_p of integers localized at the prime p. Recall

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that a submodule H of a p-local group G is *isotype* provided $p^\alpha G \cap H = p^\alpha H$ for all ordinals α, and a submodule N is *nice* in G if $(p^\alpha G + N)/N = p^\alpha(G/N)$ for all ordinals α. A submodule B is *balanced* in G provided B is both isotype and nice in G, and the corresponding exact sequence $0 \to B \to G \to G/B \to 0$ is said to be *balanced exact.* The projectives for balanced exact sequences have been recently studied by the author in [6], and the characterization given therein is crucial to our present study. Call a submodule N of G *K-nice* if N is nice in G and G/N is a *K-module,* where a module is a K-module if every finitely generated submodule is a finite extension of a finite-rank free valuated module. The height of an element $x \in G$ will be denoted by $|x|_G$ (or simply $|x|$ when G is understood). We say that a direct sum $\langle x_1 \rangle \oplus \langle x_2 \rangle$ is *free valuated* if x_1 and x_2 both have infinite order with no gaps in their height sequences and $|t_1 x_1 + t_2 x_2| = \min\{|t_1 x_1|, |t_2 x_2|\}$ for every $t_1, t_2 \in Z_p$. The latter condition is summarized by calling the direct sum a *valuated coproduct.* Of course, this concept of a valuated coproduct generalizes to a direct sum of an arbitrary number of submodules. It is shown in [6] that a module G is a balanced projective if and only if G satisfies Hill's third axiom of countability [2] with respect to K-nice submodules.

The generalization of the third axiom of countability so fundamental to [1] will be important to our study, and we shall adopt the notation and terminology of that paper. Recall that an *$H(\kappa)$-family* in a module M is a collection $\mathscr{C}$ of submodules of M such that

(H1) $0 \in \mathscr{C}$,

(H2) $\mathscr{C}$ is closed under module union, and

(H3) if $C \in \mathscr{C}$ and if A is any submodule of M of cardinality $\leqslant \kappa$, then there is a $B \in \mathscr{C}$ that contains both C and A with B/C having cardinality at most κ.

A *$G(\kappa)$-family* in M is defined analogously with property (H2) replaced by the following:

(G2) $\mathscr{C}$ is closed under unions of chains.

Finally an *$F(\kappa)$-family* in M is a well-ordered ascending chain of submodules that is continuous, beginning at 0, ending at M, with the quotient of any two adjacent members having cardinality at most κ. We will also require the notion of κ-separability. If S is any subset of G and x is any element of G, we let $|x + S| = \sup\{|x + s| : s \in S\}$, and if κ is an infinite cardinal, a submodule H of G is said to be *κ-separable* in G provided for any $x \in G$, $|x + H| = |x + S|$, for some subset S of H whose cardinality does not exceed κ.

We will begin by analyzing the techniques used in [3] to obtain a generalization of the main theorem of that paper to isotype submodules of

p-local balanced projectives. We will then have a necessary and sufficient condition for an isotype submodule of a p-local balanced projective module to be itself balanced projective. The paper will close with an application of this result to the determination of the balanced-projective dimension of an arbitrary p-local abelian group. We will then be able to discern that the balanced-projective dimension of a countable p-local group and of a p-local Warfield group is $\leqslant 1$.

1. Isotype Submodules of p-Local Balanced Projectives

As noted above, G is balanced projective if and only if G satisfies the third axiom of countability with respect to K-nice submodules; that is, G possesses an $H(\aleph_0)$-family of K-nice submodules. Observe that when G is torsion, this reduces to Hill's well-known characterization of totally projective p-groups. We need to establish the equivalence of this axiom with those of G possessing a $G(\aleph_0)$-family or an $F(\aleph_0)$-family of K-nice submodules. Hill in [2] established the equivalence among these axioms in the case of totally projective p-groups, and he relied upon this result throughout [3].

Theorem 1.1. *For a p-local module G, the axioms G has an $H(\aleph_0)$-family of K-nice submodules, G has a $G(\aleph_0)$-family of K-nice submodules, and G has an $F(\aleph_0)$-family of K-nice submodules are all equivalent.*

Proof. It is clear that if G has an $H(\aleph_0)$-family of K-nice submodules, then G has a $G(\aleph_0)$-family of K-nice submodules, and this implies that G will have an $F(\aleph_0)$-family of K-nice submodules; so we will assume that G has an $F(\aleph_0)$-family of K-nice submodules and prove that G has an $H(\aleph_0)$-family of K-nice submodules.

Given K-nice submodules $N_0 \subseteq N_1$ with N_1/N_0 countable, it is clear that there is an ascending sequence $\{M_n\}_{n<\omega}$ of submodules such that $M_0 = N_0$, $\bigcup_{n<\omega} M_n = N_1$, and M_{n+1}/M_n is cyclic of either infinite or prime order. That each M_n is itself a K-nice submodule of G follows from Corollary 1.8 in [6]. In the case where M_{n+1}/M_n is an infinite cyclic module, Corollary 1.3 in [6] will yield a valuated coproduct $M_n \oplus \langle x_n \rangle$ with $M_{n+1}/(M_n \oplus \langle x_n \rangle)$ finite and x_n having no gaps in its height sequence. Intercalculating further finite cyclic quotients between $M_n \oplus \langle x_n \rangle$ and M_{n+1}, we can assume without loss of generality that for all n, either M_{n+1}/M_n is cyclic of order p or else $M_{n+1} = M_n \oplus \langle x_n \rangle$ is a valuated coproduct. Continuing in this manner, one obtains G as the union of a smooth chain of submodules $\{M_\alpha\}_{\alpha<\mu}$ where $M_{\alpha+1} = \langle M_\alpha, x_\alpha \rangle$ such that for each α one of the following holds:

(a) $px_\alpha \in M_\alpha$ and x_α is proper with respect to M_α, or

(b) x_α has infinite order modulo M_α, $\langle x_\alpha \rangle \oplus M_\alpha$ is a valuated coproduct, and x_α has no gaps in its height sequence.

Now as in the proof of Theorem 1 in [2], we observe that $G = \langle x_\alpha \rangle_{\alpha < \mu}$ and each $0 \neq g \in G$ can be represented uniquely in the form

$$g = t_1 x_{\alpha(1)} + t_2 x_{\alpha(2)} + \cdots + t_k x_{\alpha(k)},$$

where $\alpha(1) < \alpha(2) < \cdots < \alpha(k)$ and $0 < t_i < p$ if condition (a) holds for $x_{\alpha(i)}$. We will call this the standard representation for g and denote it by (*). By the way the x_α's were chosen and since $0 < t_i < p$ when condition (a) holds for $x_{\alpha(i)}$, it is clear that $|g| = \min\{|t_i x_{\alpha(i)}|\}_{i=1,\ldots,k}$.

Let T denote the set of ordinals less than μ, and define a subset S of T to be *closed* if each nonzero element of the subgroup $N_s = \langle x_\alpha \rangle_{\alpha \in s}$ of G has a standard representation (*) based on the exhibited generators of N_s. Hill shows in [2] that the union of an arbitrary number of closed subsets of T is again closed and also that a countable subset of T has a countable closure. It is now clear that the collection $\mathscr{C} = \{N_S : S \text{ is a closed subset of } T\}$ will be an $H(\aleph_0)$-family in G, and so it only remains to show that each element in $\mathscr{C}$ is actually K-nice.

Suppose $g + N_S$ is an element of G/N_S. Write g in its standard representation (*) and choose an element $g + n \in g + N_S$ which has no generators of S in its standard representation (*). Because the property $|g| = \min\{|t_i x_{\alpha(i)}|\}_{i=1,\ldots,k}$ holds for every element of G, it follows that $g + n$ will have maximal height among the elements of $g + N_S$. Thus N_S is nice in G. Now consider $X = \{x_\alpha : x_\alpha \text{ satisfies condition (b) with respect to } M_\alpha\}$. Since obviously $\langle X \rangle$ is a free-valuated submodule of G and $G/\langle X \rangle$ is torsion, X is a so-called K-basis for G. Set $X_S = \{x_\alpha : x_\alpha \in X \text{ and } x_\alpha \notin N_S\}$. By the way the x_α's are chosen and since S is closed, it follows that $\langle X_S \rangle \cap N_S = 0$ and $\langle X_S \rangle \oplus N_S$ is a valuated coproduct. Hence the set $\{x + N_S : x \in X_S\}$ is a K-basis for G/N_S, and so G/N_S is a K-module. (The fact that any module which has a K-basis is a K-module was observed in [6].) Hence N_S is a K-nice submodule of G, and so G has an $H(\aleph_0)$-family of K-nice submodules.

THEOREM 1.2. *Suppose H is an isotype submodule of G. If H is balanced projective, then H must be $\aleph_0$-separable in G.*

Proof. Suppose that H is a balanced projective module that is isotype in G. If H is not $\aleph_0$-separable in G, then there exists $g \in G$ and a limit ordinal λ whose cofinality is not ω such that $|g + x| < \lambda$ for each $x \in H$ but $\lambda = \sup\{|g + x| : x \in H\}$. Since $H/p^\lambda H$ is balanced projective and $H/p^\lambda H = H/(p^\lambda G \cap H) \simeq (H + p^\lambda G)/p^\lambda G$ is isotype in $G/p^\lambda G$, there is no loss in generality in assuming that $p^\lambda G = 0$.

Now since H is a balanced projective of length less than or equal to λ, then by Warfield's description of the p-local balanced projectives in [9], H is the direct sum of submodules having length strictly less than λ. Hence H is complete in its λ-topology by Proposition 4 in [7]. Since $\sup\{|g+x| : x \in H\} = \lambda$ where λ is a limit ordinal, for each $\alpha < \lambda$, there exists an $h_\alpha \in H$ such that $\alpha \leqslant |g - h_\alpha| < \lambda$. Thus $\{h_\alpha\}_{\alpha<\lambda}$ yields a Cauchy net in H in the λ-topology. This can be seen since

$$h_\beta - h_\alpha = g - h_\alpha - (g - h_\beta) \in (p^\alpha G + p^\beta G) \cap H = p^\beta H$$

for all $\beta \leqslant \alpha$. Since H is complete, there exists an $h^* \in H$ such that $h^* - h_\beta \in p^\beta H$ for all $\beta < \lambda$. But then $|g - h^*| = \lambda$, and thus H is $\aleph_0$-separable in G.

Our main objective for this section is to generalize Theorem 4 in [3]. For the most part, Hill's proof carries over with but slight modification to the p-local case. There are, however, a few details that need to be straightened out first, and we will isolate these points in the following two lemmas. We will require the notion of compatibility introduced in [3]. Two submodules A and B of G are said to be *compatible*, written $A \| B$, provided for each $(a, b) \in A \times B$, there exists $x \in A \cap B$ with $|a+x| \geqslant |a+b|$.

LEMMA 1.3. *Suppose that H is a balanced projective module that appears as an isotype submodule in G which is also balanced projective. Let $\mathscr{C}_G$ and $\mathscr{C}_H$ denote $H(\aleph_0)$-families of K-nice submodules for G and H, respectively. If $B \in \mathscr{C}_G$, $B \cap H \in \mathscr{C}_H$, and $B \| H$, then both G/B and $(H+B)/B$ are balanced projective modules with $H+B/B$ isotype in G/B.*

Proof. Set $\mathscr{C}_{G/B} = \{C/B : C \in \mathscr{C}_G\}$. C/B is clearly nice in G/B since $B \in \mathscr{C}_G$, and it is also evident that C/B is K-nice in G/B since $G/B/C/B \simeq G/C$ is a K-module. Hence $\mathscr{C}_{G/B}$ is an $H(\aleph_0)$-family of K-nice submodules for G/B, which implies that G/B is balanced projective. Since $H \cap B \in \mathscr{C}_H$, the above argument can also be used to show that $H/H \cap B$ is balanced projective. It is well known that $H/H \cap B \simeq H+B/B$, and so $H+B/B$ is also balanced projective. The fact that $H+B/B$ is isotype in G/B is proved on page 318 of [3] and uses the compatibility of H and B together with the fact that H is isotype in G.

LEMMA 1.4. *If B is a K-nice submodule of G, H is an isotype submodule of G, and $H \| B$, then $A = H \cap B$ is a K-nice submodule of H.*

Proof. Hill proves in [3] that A is nice in H, and so we merely need to show that H/A is a K-module. Now

$$H/A = H/H \cap B \simeq H + B/B$$

where $H + B/B$ is isotype in G/B by Hill's proof in [3] again. It is clear that isotype submodules of K-modules are K-modules since if H is isotype in G, then the heights of elements in H computed in G are the same as the heights computed in H. Since G/B is a K-module, it follows that $H + B/B$ is a K-module, which implies that H/A is a K-module. Hence A is a K-nice submodule of H.

We are now ready to prove the main theorem of this section. If H is a submodule of G, we follow Hill in [3] by saying that G satisfies *the third axiom of countability over H with respect to $\aleph_0$-separable submodules* provided there exists a collection $\mathscr{D}$ of $\aleph_0$-separable submodules $K \supseteq H$ of G satisfying the following conditions:

(0) $H \in \mathscr{D}$,

(1) $\langle K_i \rangle_{i \in I} \in \mathscr{D}$ if $K_i \in \mathscr{D}$ for each $i \in I$, and

(2) if $H \subseteq L \subseteq G$ and L/H is countable, then there exists $K \in \mathscr{D}$ such that $K \supseteq L$ and K/H is countable.

For short, we will simply say that G has an $H(\aleph_0)$-family of $\aleph_0$-separable submodules over H when G satisfies this axiom. Hill proved in Theorem 3 in [3] the equivalence of this axiom with the following apparently weaker condition: there exists a chain of $\aleph_0$-separable submodules

$$H = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\alpha \subseteq \cdots, \qquad \alpha < \tau$$

satisfying the conditions

(i) $K_\beta = \bigcup_{\alpha < \beta} K_\alpha$ if β is a limit ordinal,

(ii) $K_{\alpha+1}/K_\alpha$ is countable, and

(iii) $G = \bigcup_{\alpha < \tau} K_\alpha$.

When G satisfies this latter condition, we will say that G has an $F(\aleph_0)$-family of $\aleph_0$-separable submodules over H.

THEOREM 1.5. *Suppose that H is an isotype submodule of the balanced projective module G. Then H is also balanced projective if and only if G satisfies the third axiom of countability over H with respect to $\aleph_0$-separable submodules.*

Proof. First assume that H is balanced projective. Let $\mathscr{C}_G$ and $\mathscr{C}_H$ denote $H(\aleph_0)$-families of K-nice submodules of G and H, respectively. Since H is balanced projective, note that H is $\aleph_0$-separable in G by Theorem 1.2. Suppose that

$$0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots, \qquad \alpha < \gamma$$

is an ascending chain of submodules of G that satisfies the following five conditions for $\alpha<\gamma$:

(1) $B_\alpha \in \mathscr{C}_G$,

(2) $B_\alpha \cap H \in \mathscr{C}_H$,

(3) $B_\alpha \parallel H$,

(4) the cardinality of $B_{\alpha+1}/B_\alpha \leqslant \aleph_0$ whenever $\alpha+1\leqslant\gamma$, and

(5) $B_\beta=\bigcup_{\alpha<\beta} B_\alpha$ whenever $\beta<\gamma$ is a limit ordinal.

Hill's constructions in his proof of Theorem 4 in [3] will now carry over routinely, using the fact that for each α, both G/B_α and $H+B_\alpha/B_\alpha$ are balanced projectives with $H+B_\alpha/B_\alpha$ isotype in G/B_α (by Lemma 1.3), to conclude that this sequence can be extended up to G while maintaining the properties (1)–(5). With this chain, we set $K_\alpha=H+B_\alpha$ and consider the chain

$$H=K_0\subseteq K_1\subseteq\cdots\subseteq K_\alpha\subseteq\cdots\subseteq G.$$

This chain will be an $F(\aleph_0)$-family of $\aleph_0$-separable submodules for G over H provided each K_α is $\aleph_0$-separable in G. So suppose that $|g+K_\alpha|=\mu$ where $\operatorname{cof}(\mu)>\omega$. We observe that $\mu=|g+H+B_\alpha|\leqslant |g+B_\alpha+(H+B_\alpha/B_\alpha)|$. Since $H+B_\alpha/B_\alpha$ is isotype in G/B_α and $H+B_\alpha/B_\alpha$ is balanced projective, it follows that $H+B_\alpha/B_\alpha$ is $\aleph_0$-separable in G/B_α by Theorem 1.2. Hence by Proposition 1 in [3], there must be some $h\in H$ such that $\mu\leqslant|g+h+B_\alpha|_{G/B_\alpha}$. Since B_α is nice in G, there exists some $b\in B_\alpha$ such that $\mu\leqslant|g+h+b|_G$, and thus K_α is $\aleph_0$-separable in G by Proposition 1 in [3]. Hence G has an $F(\aleph_0)$-family of $\aleph_0$-separable submodules over H, and so G satisfies the third axiom of countability over H with respect to $\aleph_0$-separable submodules by Theorem 3 in [3].

Conversely, suppose that G satisfies the third axiom of countability over H with respect to $\aleph_0$-separable submodules, G is balanced projective, and H is isotype in G. Let $\mathscr{C}$ be an $H(\aleph_0)$-family of *K-nice* submodules of G, and let $\mathscr{D}$ be a collection of $\aleph_0$-*separable* submodules of G satisfying the third axiom of countability over H. Let us examine the consequences of a subgroup B of G satisfying the following three conditions:

(i) $B\in\mathscr{C}$,

(ii) $H+B\in\mathscr{D}$, and

(iii) $B\parallel H$.

Again, Hill's constructions in his proof of Theorem 4 in [3] will carry over routinely to guarantee the existence of an ascending chain

$$0=B_0\subseteq B_1\subseteq\cdots\subseteq B_\alpha\subseteq\cdots$$

of submodules of G satisfying conditions (i)-(iii) with $B_{\alpha+1}/B_\alpha$ countable, $B_\beta = \bigcup_{\alpha<\beta} B_\alpha$ when β is a limit ordinal, and G the union of all the B_α's. Set $A_\alpha = H \cap B_\alpha$, and consider the corresponding chain

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots \subseteq H$$

of submodules of H. This chain will be an $F(\aleph_0)$-family of K-nice submodules of H provided each A_α is K-nice in H. But this is now immediate by Lemma 1.4. Hence H has an $H(\aleph_0)$-family of K-nice submodules by Theorem 1.1, and so H is a balanced projective module.

COROLLARY 1.6. *Suppose that H is an isotype submodule of a balanced projective module G such that the cardinality of G does not exceed $\aleph_1$. Then a necessary and sufficient condition for H to be balanced projective is that H is $\aleph_0$-separable in G.*

Proof. If H is balanced projective, then H is necessarily $\aleph_0$-separable in G by Theorem 1.2. Conversely, suppose H is $\aleph_0$-separable in G. Since the cardinality of $G \leqslant \aleph_1$, there exists an ascending continuous chain

$$H = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots \subseteq G,$$

where for each α, G_α/H is countable. By Proposition 2.5 in [1], each G_α is $\aleph_0$-separable in G, and so we may invoke Theorem 3 in [3] to prove that G must satisfy the third axiom of countability over H with respect to $\aleph_0$-separable submodules. Hence H is balanced projective by Theorem 1.5.

For balanced submodules, the conditions of Theorem 1.5 can be formulated more simply. By Proposition 2.4 in [1], if H is nice in G and $H \subseteq K \subseteq G$, then K is κ-separable in G if and only if K/H is κ-separable in G/H. Therefore, as an immediate corollary of 1.5, we have the following observation.

COROLLARY 1.7. *Suppose that H is a balanced submodule of the p-local balanced projective group G. Then H is also balanced projective if and only if G/H contains an $H(\aleph_0)$-family of $\aleph_0$-separable submodules.*

2. THE BALANCED-PROJECTIVE DIMENSION

In this section we will discuss the balanced-projective dimension of an arbitrary p-local group. Motivated by the results in [1], we will be able to characterize all the modules which have balanced-projective dimension n, where n is a nonnegative integer. The tools used to characterize the balanced-projective dimension of p-local modules having elements of infinite order will be precisely the same as those used in [1]. We will thus require the notion of an Axiom 3: κ module as in Definition 2.3 in [1]. Our definition will be adjusted slightly so that when $\kappa = \aleph_{-1}$ is finite, an Axiom 3: κ module will be balanced projective. We will also soon need the notion of the coset valuation. Suppose N is a submodule of G and let $x \in G$. The *coset valuation* of $x+N$ in G/N is denoted $\|x+N\|$ and defined by $\|x+N\| = \sup\{|x+n|+1 : n \in N\}$.

DEFINITION 2.1. Let $\kappa = \aleph_\alpha$ denote an infinite cardinal number. A module M satisfies *Axiom* 3: κ (and M is said to be an *Axiom* 3: κ *module*) if M has an $H(\kappa)$-family of κ-separable submodules. If $\kappa = \aleph_{-1}$ is finite, then M satisfies Axiom 3: κ provided M has an $H(\aleph_0)$-family of K-nice submodules.

THEOREM 2.2. *For a p-local module G, the axioms G is an Axiom* 3: κ *module, G has a* $G(\kappa)$*-family of* κ*-separable submodules, and G has an* $F(\kappa)$*-family of* κ*-separable submodules are all equivalent.*

Proof. The case when $\kappa = \aleph_{-1}$ is finite was handled in Theorem 1.1; so we will assume κ is an infinite cardinal. As in the proof of Theorem 3.2 in [1], it suffices to construct an $H(\kappa)$-family of κ-separable submodules from an $F(\kappa)$-family of κ-separable submodules. Given an $F(\kappa)$-family of κ-separable submodules, construct a *composition series of* κ*-separable submodules* for G; that is, an ascending chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots, \qquad \alpha < \tau$$

with $G = \bigcup_{\alpha<\tau} N_\alpha$, each N_α κ-separable in G, and $N_{\alpha+1}/N_\alpha$ cyclic of infinite or prime order. For each α such that $\alpha+1<\tau$, we let $N_{\alpha+1} = \langle N_\alpha, x_\alpha \rangle$ where either $px_\alpha \in N_\alpha$, or else x_α has infinite order modulo N_α. Hence $N_\beta = \langle x_\alpha : \alpha < \beta \rangle$, and if $\mathrm{cof}(\|x_\alpha + N_\alpha\|) = 0$, then we can choose x_α so that $|x_\alpha| = |x_\alpha + N_\alpha|$. Once the x_α's have been so selected, we observe that each element $x \in G$ can be represented uniquely as

$$x = c_0 x_{\alpha(0)} + c_1 x_{\alpha(1)} + \cdots + c_n x_{\alpha(n)},$$

where each $c_i \in Z_p$, $1 \leqslant c_i < p$ if $px_{\alpha(i)} \in N_{\alpha(i)}$, and $\alpha(0) < \alpha(1) < \cdots < \alpha(n)$. This representation is called the standard representation for x.

Now let $T = \tau$, the index for the composition series of G and define a subset S of T to be *closed* if it satisfies the following two properties:

(a) if $\alpha \in S$, then the standard representation of px_α involves no elements of T outside of S, and

(b) if $\alpha \in S$ and we define $N_\alpha(S) = \langle x_\beta : \beta \in S, \beta < \alpha \rangle$, then $\|kx_\alpha + N_\alpha\| = \|kx_\alpha + N_\alpha(S)\|$ for $k = 1$ if $px_\alpha \in N_\alpha$ and for all $k \in Z_p$ if $px_\alpha \notin N_\alpha$.

It is clear that $N_\alpha(S) \subseteq N_\alpha$ and thus the equality in (b) is the same as the inequality $\leqslant$. Observe that the union of any number of closed sets is again closed, and the back-and-forth technique can be used to prove that if R is any subset of T of cardinality not exceeding κ, then there exists a closed subset S of T containing R of cardinality not exceeding κ. (See the argument in [1].) We define $\mathscr{C}$ to be the collection of submodules of G such that $N \in \mathscr{C}$ if and only if, for some closed subset S of T, $N = \langle x_\alpha : \alpha \in S \rangle$. To prove that $\mathscr{C}$ is an $H(\kappa)$-family of κ-separable submodules, it is enough to prove that each $N \in \mathscr{C}$ is κ-separable.

Let $N = G(S) = \langle x_\alpha : \alpha \in S \rangle$ where S is a closed subset of T and assume that N is not κ separable in G. By Proposition 2.2 in [1], we must have $\operatorname{cof}(\|y + N\|) > \kappa$ for some $y \in G$. Among all such choices of y, choose one that produces a minimal $\alpha(n)$ in its standard representation

$$y = c_0 x_{\alpha(0)} + c_1 x_{\alpha(1)} + \cdots + c_n x_{\alpha(n)}.$$

It is clear that $\alpha(n) \notin S$ and if $\|y + N\| = \mu$, then $|x_{\alpha(n)}| < \mu$. Suppose $\lambda = \|x_{\alpha(n)} + N_{\alpha(n)}\|$. Since $\operatorname{cof}(\mu) > \kappa$ and $N_{\alpha(n)}$ is κ-separable in G, we know that $\lambda \neq \mu$. If $\lambda > \mu$, then $\|x_{\alpha(n)} + N_{\alpha(n)}\| \geqslant \mu + 1$ implies that $|x_{\alpha(n)} + z| \geqslant \mu$ for some $z \in N_{\alpha(n)}$. Hence, if $y' = y - (c_n x_{\alpha(n)} + c_n z)$, then $\|y' + N\| = \|y + N\| = \mu$. But $y' \in N_{\alpha(n)}$, and therefore its standard representation involves only α's less than $\alpha(n)$, which contradicts the choice of y. Thus we must have $\lambda < \mu$. Since $\alpha(n) \notin S$ and S is closed in T, no element belonging to N uses the generator $x_{\alpha(n)}$ in its standard representation. Thus

$$\|y + N_{\alpha(n)}\| = \|x_{\alpha(n)} + N_{\alpha(n)}\| = \lambda \geqslant \|y + (N \cap N_{\alpha(n)+1})\|$$

since $N \cap N_{\alpha(n)+1} \subseteq N_{\alpha(n)}$. Since $\lambda < \mu$, there must exist $w \in N$ such that $|y + w| \geqslant \lambda$ and $w \notin N_{\alpha(n)+1}$. Among all the possible choices for w, choose one that produces a minimal $\beta(m)$, where the standard representation of w is

$$w = d_0 x_{\beta(0)} + d_1 x_{\beta(1)} + \cdots + d_m x_{\beta(m)},$$

with $1 \leqslant d_i < p$ when $px_{\beta(i)} \in N_{\beta(i)}$ and $\beta(0) < \beta(1) < \cdots < \beta(m)$. Note that $\beta(m) > \alpha(n)$. Since $|y + w| \geqslant \lambda$, we must conclude that $\|d_m x_{\beta(m)} + N_{\beta(m)}\| \geqslant \lambda + 1$ and consequently, $\|d_m x_{\beta(m)} + N_{\beta(m)}(S)\| \geqslant \lambda + 1$. But then $|d_m x_{\beta(m)} + v| \geqslant \lambda$ for some $v \in N_{\beta(m)}(S)$, which will imply that $w' = w - (d_m x_{\beta(m)} + v) \in N_{\beta(m)}(S)$ and $|y + w'| \geqslant \lambda$. This is a contradiction to the choice of w and $\beta(m)$. Therefore, N must be κ-separable in G, and $\mathscr{C}$ will be an $H(\kappa)$-family of κ-separable submodules for G.

Suppose there is an exact sequence

$$\cdots \to T_n \xrightarrow{\delta_n} T_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \to 0,$$

where each T_i is balanced projective and the image of δ_i is balanced in T_{i-1}. Then we define the *balanced-projective dimension* (b.p.d.) of M to be n if n is the smallest index $\geqslant 0$ with $\mathrm{Im}(\delta_n)$ balanced projective. The b.p.d. of a module is well defined and b.p.d. $M=0$ if and only if M is balanced projective. Recall that for an infinite cardinal κ, the cardinal number κ^+ is the smallest cardinal greater than κ, and when κ is finite, κ^+ is understood to be $\aleph_0$. The following theorems can be proved almost verbatim from their torsion counterparts in [1] by using Corollary 1.7 in the case when κ is finite.

THEOREM 2.3. *Let $0 \to B \to T \to M \to 0$ be a balanced exact sequence, where T is balanced projective. If M satisfies Axiom 3: κ^+, then B must satisfy Axiom 3: κ, and if B satisfies Axiom 3: κ, then M must satisfy Axiom 3: κ^+.*

Proof. We will suppose M satisfies Axiom 3: κ^+ and prove that B satisfies Axiom 3: κ. If $\kappa = \aleph_{-1}$ is finite, then $\kappa^+ = \aleph_0$, and the result follows by Corollary 1.7. Thus we will assume that κ is infinite. Let $\mathscr{C}_T$ and $\mathscr{C}_M$ denote $H(\kappa^+)$-families of K-nice submodules of T and κ^+-separable submodules of M, respectively. By Lemma 1.4 in [1], we may assume without loss of generality that $\mathscr{C}_M = \{\langle N, B\rangle/B : N \in \mathscr{C}_T\}$. By Lemma 4.1 in [1], there is a $G(\kappa^+)$-subfamily $\mathscr{C}'_T$ of $\mathscr{C}_T$ such that $N \| B$ for each $N \in \mathscr{C}'_T$. Let $\mathscr{D}_T$ be an $F(\kappa^+)$-subfamily of $\mathscr{C}'_T$. The proof of Theorem 4.2 in [1] will now carry over routinely to show that $\mathscr{C}_B = \{B \cap N : N \in \mathscr{D}_T\}$ is an $F(\kappa^+)$-family of nice submodules of B, and one can then refine $\mathscr{C}_B$ to an $F(\kappa)$-family of κ-separable submodules by using Proposition 2.5 in [1]. Then B satisfies Axiom 3: κ by Theorem 2.2.

Conversely, we will assume that B satisfies Axiom 3: κ and prove M must satisfy Axiom 3: κ^+. The result is proved in Corollary 1.7 in the case where κ is finite, and so we will again assume that κ is an infinite cardinal. Let $\mathscr{C}_B$ and $\mathscr{C}_T$ denote $G(\kappa)$-families of κ-separable submodules of B and K-nice submodules of T, respectively. By Lemma 1.6 in [1], there is an $F(\kappa)$-subfamily $\mathscr{C}'_B$ of $\mathscr{C}_B$ and a $G(\kappa)$-subfamily $\mathscr{C}'_T$ of $\mathscr{C}_T$ such that $\mathscr{C}'_B \subseteq B \cap \mathscr{C}'_T \subseteq \mathscr{C}_B$. By Lemma 4.3 in [1], there exists a $G(\kappa^+)$-subfamily $\mathscr{C}''_T$ of $\mathscr{C}'_T$ such that $N \| B$ for each $N \in \mathscr{C}''_T$. Now let $\mathscr{D}_T$ be an $F(\kappa^+)$-subfamily of $\mathscr{C}''_T$, and define $\mathscr{C}_M = \{\langle N, B\rangle/B : N \in \mathscr{D}_T\}$. The argument given by Fuchs and Hill in Theorem 4.4 of [1] will now apply to prove that $\mathscr{C}_M$ is an $F(\kappa^+)$-family of κ^+-separable submodules. Hence M satisfies Axiom 3: κ^+ by Theorem 2.2.

THEOREM 2.4. *For each* $n \geqslant 0$, b.p.d. $M \leqslant n$ *if and only if* M *satisfies Axiom* 3: $\aleph_{n-1}$.

Proof. We will prove this theorem by induction on the integer n. When $n = 0$, then M satisfies Axiom 3: $\aleph_{-1}$ if and only if M is balanced projective, which implies that b.p.d. $M = 0$. Now suppose that $n \geqslant 1$ and that $0 \to B \to T \to M \to 0$ is a balanced exact sequence where T is balanced projective. First assume b.p.d. $M \leqslant n$. Then b.p.d. $B \leqslant n-1$, and the induction hypothesis implies that B satisfies Axiom 3: $\aleph_{n-2}$. Theorem 2.3 will now imply that M satisfies Axiom 3: $\aleph_{n-1}$. Conversely, if M satisfies Axiom 3: $\aleph_{n-1}$, then Theorem 2.3 can be applied again to prove that B must satisfy Axiom 3: $\aleph_{n-2}$. The induction hypothesis then says that b.p.d. $B \leqslant n-1$, and so b.p.d. $M \leqslant n$.

COROLLARY 2.5. *If* M *is a* p*-local Warfield module or* M *is countable, then* b.p.d. $M \leqslant 1$.

Proof. Suppose M is a p-local Warfield module. Theorem 4.1 in [5] states that M must satisfy the third axiom of countability with respect to *knice submodules*. A knice submodule is a nice submodule together with a property concerning the elements of infinite order. The important point is that M has an $H(\aleph_0)$-family of nice submodules, which implies that b.p.d. $M \leqslant 1$ by Theorem 2.4. If M is a countable p-local module, then clearly M has an $H(\aleph_0)$-family of $\aleph_0$-separable submodules (namely 0 and M), and so b.p.d. $M \leqslant 1$.

Nongxa exhibited in [8] a countable abelian group of b.p.d. >1. Moreover, Nongxa only needed to consider finite rank torsion-free groups to produce his example. Hence the assertion in Corollary 2.5 about countable groups is false in the global setting, although the question of whether global Warfield groups have b.p.d. $\leqslant 1$ is open. At any rate, it is clear that there are formidable difficulties in characterizing the global groups of finite b.p.d. Still it is possible that the techniques of this paper can be generalized to treat certain special classes of global groups such as Warfield groups or separable torsion free groups.

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